

# MARKOV PROCESSES WITH FREE-MEIXNER LAWS

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**ABSTRACT.** We study a time-non-homogeneous Markov process which arose from free probability, and which also appeared in the study of stochastic processes with linear regressions and quadratic conditional variances. Our main result is the explicit expression for the generator of the (non-homogeneous) transition operator acting on functions that extend analytically to complex domain.

The paper is self-contained and does not use free probability techniques.

## 1. INTRODUCTION

In this paper we study a special class of (non-homogeneous) Markov processes whose univariate law form a semigroup with respect to the so called free additive convolution of measures. These processes arise as the "classical versions" of the corresponding non-commutative free-Lévy processes in the sense that their time-ordered moments coincide, see Biane (1998, page 144). The same class of Markov processes also appeared as one of the examples in the study of "quadratic harnesses", i.e. processes with linear regression and quadratic conditional variances under double-sided conditioning with respect to past and future. The paper however is self-contained and does not rely on free probability techniques or "quadratic harnesses", except for motivation or "inspiration". (For example, the expression for the martingale in Proposition 2.2 came from papers of Biane and Anshelevich but in this paper we verify the martingale property by direct integration.) To avoid distracting the reader, motivation and connections with free probability and with "quadratic harnesses" are discussed in a separate section at the end of the paper.

The paper is organized as follows. In Section 2 we define the family of Markov processes and state our main results. Section 3 collects elementary integrals needed for the proofs. The integrals are then used in the proofs of the main results in Section 4. In A we discuss relations to previous results, including connections to free probability.

## 2. RESULTS

We consider a family of probability measures  $\{P_{s,t}(x, dy) : 0 \leq s < t, x \in \mathbb{R}\}$  on Borel sets of the real line which depend on two auxiliary parameters  $\theta \in \mathbb{R}$  and  $\tau \geq 0$ . The definition is somewhat cumbersome due to the possible presence of an

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atom which may occur at the points that are given parametrically as

$$(2.1) \quad a_*(t) = \begin{cases} -t/\theta & \text{if } \tau = 0, \theta \neq 0, \\ -t \frac{\theta - \sqrt{\theta^2 - 4\tau}}{2\tau} & \text{if } \tau > 0, \theta > 0, \\ -t \frac{\theta + \sqrt{\theta^2 - 4\tau}}{2\tau} & \text{if } \tau > 0, \theta < 0. \end{cases}$$

Probability measures  $P_{s,t}(x, dy)$  are specified by their absolutely continuous component and discrete components (there is no singular component). The continuous component is given by the density

$$(2.2) \quad \frac{1}{2\pi} \frac{(t-s)\sqrt{4(t+\tau) - (y-\theta)^2}}{\tau(y-x)^2 + \theta(t-s)(y-x) + tx^2 + sy^2 - (s+t)xy + (t-s)^2},$$

supported on  $y$  from the interval  $[\theta - 2\sqrt{t+\tau}, \theta + 2\sqrt{t+\tau}]$ . The discrete component of  $P_{s,t}(x, dy)$  is zero except for the following cases.

- (1) If  $\tau = 0$ ,  $\theta \neq 0$ , and  $x = a_*(s) = -s/\theta$ , then with  $b^+ = \max\{b, 0\}$  the discrete part of  $P_{s,t}(x, dy)$  is given by

$$\frac{(1 - t/\theta^2)^+}{1 - s/\theta^2} \delta_{a_*(t)}.$$

In particular, the discrete component is absent for  $t \geq \theta^2$ .

- (2) If  $\tau > 0$ ,  $\theta^2 > 4\tau$  and  $x = a_*(s)$ , then the discrete part of  $P_{s,t}(x, dy)$  is given by

$$\frac{\left(1 - \frac{t}{2\tau} \frac{|\theta| - \sqrt{\theta^2 - 4\tau}}{\sqrt{\theta^2 - 4\tau}}\right)^+}{1 - \frac{s}{2\tau} \frac{|\theta| - \sqrt{\theta^2 - 4\tau}}{\sqrt{\theta^2 - 4\tau}}} \delta_{a_*(t)}.$$

In particular, the discrete component is absent for  $t \geq 2\tau\sqrt{\theta^2 - 4\tau}/(|\theta| - \sqrt{\theta^2 - 4\tau})$ .

The laws  $P_{0,t}(0, dy)$  are the free Meixner laws in Note A.1.

Family  $\{P_{s,t}(x, dy) : 0 \leq s < t, x \in \mathbb{R}\}$  forms transition probabilities of a Markov process. This fact is implicit in Biane (1998), and explicit in (Bryc and Wołowski, 2005, Theorem 4.3). Here we give a different proof based on the integral transform in Lemma 3.5.

**Proposition 2.1.** *For every  $\theta \in \mathbb{R}$  and  $\tau \geq 0$ , there exists a right-continuous with left limits (cadlag) Markov process  $(X_t : t \geq 0)$  with state space  $\mathbb{R}$ , initial state  $X_0 = 0$ , and such that for  $0 \leq s < t$ ,  $\Pr(X_t \in U | X_s) = P_{s,t}(X_s, U)$  with probability one.*

The univariate laws of  $X_t$  are  $P_{0,t}(0, dy)$ ; these are the free-Meixner laws in the title of the paper, see Note A.1.

Next we describe a class of martingales associated with Markov process  $(X_t)$ . We introduce the natural filtration  $\mathcal{F}_t := \sigma(X_s : s \leq t)$ ,  $t \geq 0$ .

**Proposition 2.2.** *Fix  $z \in \mathbb{C}$  such that  $\tau|z|^2 < 1$ . If  $(X_t : t \geq 0)$  is the Markov process introduced in Proposition 2.1, then the complex-valued process*

$$(2.3) \quad M_t = \frac{1}{1 - z(X_t - \theta) + (t + \tau)z^2}$$

*is an  $\mathcal{F}_t$ -martingale for  $0 \leq t < 1/|z|^2 - \tau$ .*

It might be worth pointing out that  $(M_t)$  is not a martingale for  $t > 1/|z|^2 - \tau$ , as then

$$\mathbb{E}(M_t) = \frac{t + \tau}{(t + \tau)^2 z^2 + \theta z(t + \tau) + \tau}$$

depends on  $t$ . See also Note A.3.

To state our next result we need additional notation. By  $w_{m,\sigma^2}$  we denote the Wigner's semicircle law of mean  $m$  and variance  $\sigma^2 > 0$ , given by the density

$$(2.4) \quad w_{m,\sigma^2}(dx) = \frac{\sqrt{4\sigma^2 - (x - m)^2}}{2\pi\sigma^2} 1_{|x-m| \leq 2\sigma}(x)dx.$$

For  $t > 0$ , we consider the "generator"

$$L_t(f)(x) = \lim_{h \rightarrow 0^+} \int \frac{f(y) - f(x)}{h} P_{t,t+h}(x, dy),$$

defined on bounded measurable functions  $f$  such that the limit exists. Our goal is to derive the expression for  $L_t(f)$  when  $f$  belongs to a certain family  $\mathcal{A}_t$  which contains all functions that extend analytically to the entire complex plain  $\mathbb{C}$ .

To define this family  $\mathcal{A}_t$ , we denote by  $r_t$  the radius of the disk centered at  $\theta$  that contains the support of  $X_t$ . Depending of the values of parameter  $t, \theta, \tau$ , this radius is the larger of the expressions  $2\sqrt{t}$  or  $|\theta + t/\theta|$  when  $\tau = 0$  or the larger of  $2\sqrt{t + \tau}$  and  $((t + 2\tau)|\theta| + t\sqrt{\theta^2 - 4\tau})/(2\tau)$  when  $\tau > 0$ , see (2.1). Then  $f \in \mathcal{A}_t$  if there is  $\delta > 0$  such that  $z \mapsto f(z)$  is analytic in the disk  $|z - \theta| < \frac{5}{4}(r_t + \delta)$ .

We now state our main result.

**Theorem 2.3.** *Fix  $t > 0$ . If  $f \in \mathcal{A}_t$ , then for  $x \in \text{supp}(X_t)$ ,*

$$(2.5) \quad (L_t f)(x) = \frac{\partial}{\partial x} \int_{\mathbb{R}} \frac{f(y) - f(x)}{y - x} w_{\theta, t+\tau}(dy).$$

We remark that (2.5) can be viewed as an analog of "Ito's formula" for instantaneous functions: if  $f$  is analytic in  $\mathbb{C}$  then

$$f(X_t) - \int_0^t L_s(f)(X_s) ds$$

is a martingale with respect to  $(\mathcal{F}_t)$ . We also remark that at an atom of  $X_t$  one should take the derivative before evaluating (2.5) at  $x = a_*(t)$ . Equivalently,

$$(L_t f)(x) = \int_{\mathbb{R}} \frac{f(y) - f(x) - (y - x)f'(x)}{(y - x)^2} w_{\theta, t+\tau}(dy).$$

We do not know the generators for Markov processes that correspond to more general free-Lévy processes; we also do not know the generators for the  $q$ -Meixner processes in Bryc and Wołowski (2005) when  $q \neq 0, \pm 1$ .

### 3. ELEMENTARY INTEGRALS AND AN AUXILIARY MARKOV PROCESS

For complex  $a_1, a_2, a_3, a_4$  let

$$\tilde{f}(x; a_1, a_2, a_3, a_4) = \frac{\sqrt{1 - x^2}}{(1 + a_1^2 - 2a_1x)(1 + a_2^2 - 2a_2x)(1 + a_3^2 - 2a_3x)(1 + a_4^2 - 2a_4x)}.$$

**Lemma 3.1.** *If  $|a_1|, \dots, |a_4| < 1$ , then*

$$(3.1) \quad \int_{-1}^1 \tilde{f}(x; a_1, a_2, a_3, a_4) dx = K(a_1, a_2, a_3, a_4),$$

where

$$(3.2) \quad K(a_1, a_2, a_3) = \frac{\pi}{2}(1 - a_1 a_2 a_3 a_4) \prod_{1 \leq i < j \leq 4} (1 - a_i a_j)^{-1}.$$

*Proof.* This integral is known (see Note A.7), but assuming  $a_1, \dots, a_4$  are all distinct we provide the main steps of evaluation for completeness. By partial fractions decomposition, we only need to integrate four expressions of the form

$$\frac{a_1^3}{\prod_{j=2}^4 [(a_1 - a_j)(1 - a_1 a_j)]} \frac{\sqrt{1 - y^2}}{(1 + a_1^2 - 2a_1 y)}.$$

Substituting  $y = \cos \alpha$  and using the fact that  $|a| < 1$  we get

$$\begin{aligned} \int_{-1}^1 \frac{\sqrt{1 - y^2}}{1 + a^2 - 2ay} dy &= \frac{1}{2} \int_0^{2\pi} \frac{\sin^2 \alpha}{(1 - ae^{i\alpha})(1 - ae^{-i\alpha})} d\alpha \\ &= \frac{i}{8} \oint_{|z|=1} \frac{(z^2 - 1)^2}{(1 - az)(z - a)z^2} dz = \frac{\pi}{4} \left(1 - \frac{1}{a^2}\right) + \frac{\pi}{4} \left(1 + \frac{1}{a^2}\right) = \pi/2, \end{aligned}$$

with the last integral evaluated by residua at  $z = a$  and  $z = 0$ . (The third singularity at  $z = 1/a$  is outside of the unit disk.) Summing the four expressions from the partial fractions decomposition we get (3.2).  $\square$

In general, the integral in (3.1) diverges when the parameters are on the unit circle; but there are two exceptions that arise from cancellations with the roots of  $\sqrt{1 - x^2}$ : one parameter can take one of the values  $\pm 1$  or a pair  $(a_i, a_j)$  of parameters can take the value  $(-1, 1)$ . In these two exceptional cases the integral is still given by (3.2) still holds, as can be seen by taking the limits.

The integral in (3.1) converges also if some of the parameters are outside of the unit disk. Since  $1 + a^2 - 2ax = a^2(1 + 1/a^2 - 2x/a)$ , formula (3.1) can be used to evaluate such an integral. For example, if  $|a_2|, |a_3|, |a_4| < 1$  and  $|a_1| > 1$ , then

$$(3.3) \quad \int_{-1}^1 \tilde{f}(x; a_1, a_2, a_3, a_4) dx = K(1/a_1, a_2, a_3, a_4)/a_1^2,$$

with

$$(3.4) \quad \frac{K(1/a_1, a_2, a_3, a_4)}{a_1^2} = \frac{\pi(a_1 - a_2 a_3 a_4)}{2(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)(1 - a_2 a_3)(1 - a_2 a_4)(1 - a_3 a_4)}.$$

**3.1. Probability measures.** We now introduce a two-parameter family of probability measures with parameters that satisfy the following.

**Assumption 3.2.** Let  $a_1, a_2$  be either real or complex conjugate, such that their product satisfies  $a_1 a_2 < 1$ .

Assumption 3.2 is a concise way of stating that either  $a_1 = \bar{a}_2$  are from the unit disk of the complex plane, or  $a_1, a_2$  are real and at least one of them is in the interval  $(-1, 1)$ , or if both are real but outside of  $(-1, 1)$  then they have opposite signs. We will need to consider these cases separately in the definitions and in the proofs.

Under Assumption 3.2,  $\tilde{f}(y; a_1, a_2, 0, 0)$  is real-valued, positive, and integrable. To confirm this, we need to consider separately the case when  $a_1 = \bar{a}_2$ , and the case when  $a_1, a_2$  are real. To see positivity for real  $a_1, a_2$ , we write

$$(1 + a_1^2 - 2a_1y)(1 + a_2^2 - 2a_2y) = |(1 - a_1e^{i\alpha_y})(1 - a_2e^{i\alpha_y})|^2$$

with  $\alpha_y = \arccos y$ .

The corresponding normalizing constant

$$k(a_1, a_2) = K(a_1, a_2, 0, 0) = \frac{\pi}{2(1 - a_1a_2)}$$

is well defined and positive. We therefore introduce the non-negative function

$$(3.5) \quad f(y; a_1, a_2) = \frac{1}{k(a_1, a_2)} \tilde{f}(y; a_1, a_2, 0, 0) 1_{[-1, 1]}(y).$$

By (3.1),  $f$  is a probability density function when  $|a_1|, |a_2| < 1$ . For other values of admissible parameters, it is easy to check that  $f(y)dy$  is a sub-probability measure. Adding the missing mass as the weight of (carefully selected!) atoms, we consider the following two-parameter family of probability measures:

$$(3.6) \quad \nu(dy; a_1, a_2) = \begin{cases} f(y; a_1, a_2) dy & \text{if } |a_1|, |a_2| < 1, \\ \frac{(1 \mp a_2)\sqrt{1 \pm x}}{\pi\sqrt{1 \mp x}(1 + a_2^2 - 2a_2x)} & \text{if } -1 < a_2 < 1, a_1 = \pm 1, \\ \frac{1}{\pi\sqrt{1 - x^2}} & \text{if } a_1 = \pm 1, a_2 = -a_1, \\ f(y; a_1, a_2) dy + w(a_1, a_2)\delta_{y(a_1)} & \text{if } -1 < a_2 < 1, |a_1| > 1, \\ f(y; a_1, a_2) dy + w(a_1, a_2)\delta_{y(a_1)} + w(a_2, a_1)\delta_{y(a_2)} & \text{if } a_1 > 1 \text{ and } a_2 < -1, \end{cases}$$

where the locations of the atoms are  $y(a) = (a + 1/a)/2$  and the weights of the atoms are

$$(3.7) \quad w(a, b) = \frac{a^2 - 1}{a^2 - ab}.$$

It is straightforward to verify that  $0 < w(a_1, a_2) < 1$  and that

$$w(a_1, a_2) = 1 - \frac{k(1/a_1, a_2)}{a_1^2 k(a_1, a_2)} = 1 - \int_{-1}^1 f(x; a_1, a_2) dx$$

when  $a_1, a_2$  are real,  $a_1a_2 < 1$ ,  $-1 < a_2 < 1$  and  $|a_1| > 1$ . Furthermore, it is clear that  $w(a_1, a_2), w(a_2, a_1) > 0$  and that

$$w(a_1, a_2) + w(a_2, a_1) = 1 + \frac{1}{a_1a_2} = 1 - \frac{k(1/a_1, 1/a_2)}{a_1^2 a_2^2 k(a_1, a_2)} = 1 - \int_{-1}^1 f(x; a_1, a_2) dx$$

when  $a_1 > 1, a_2 < -1$ .

We extend the definition (3.6) to the entire range of admissible parameters  $a_1, a_2$  by symmetry: we request that  $\nu(dy; a_1, a_2) = \nu(dy; a_2, a_1)$  also in all cases omitted from (3.6).

We note the following elementary formulas.

**Proposition 3.3.** *The mean of  $\nu(dy; a_1, a_2)$  is*

$$m = \int_{\mathbb{R}} y \nu(dy; a_1, a_2) = (a_1 + a_2)/2,$$

and the variance is

$$\int_{\mathbb{R}} (y - m)^2 \nu(dy; a_1, a_2) = (1 - a_1 a_2)/4.$$

For  $|z| < 1$ ,

$$(3.8) \quad \int_{\mathbb{R}} \frac{1}{1 + z^2 - 2zy} \nu(dy; a_1, a_2) = \frac{1}{(1 - a_1 z)(1 - a_2 z)}.$$

*Proof.* To compute the moments we take the derivatives of both sides of (3.8) at  $z = 0$ .

To derive formula (3.8) we need to consider separately each case that appears in (3.6). In each case we apply (3.1) to evaluate the integral over the absolutely continuous component of the measure, and add the corresponding contribution of the discrete component.

In the case  $|a_1|, |a_2| < 1$ , the left hand side of (3.8) is  $K(a_1, a_2, z, 0)/K(a_1, a_2, 0, 0)$ . From (3.2) we get (3.8).

In the case  $|a_1| > 1, |a_2| < 1$  we use (3.3). From the continuous part we get

$$\frac{K(1/a_1, a_2, z, 0)}{a_1^2 K(a_1, a_2, 0, 0)} = \frac{1 - a_1 a_2}{(a_1 - a_2)(1 - a_2 z)(a_1 - z)}.$$

The discrete part contributes

$$\frac{w(a_1, a_2)}{1 + z^2 - 2zy(a_1)} = \frac{w(a_1, a_2)}{(1 - za_1)(1 - z/a_1)} = \frac{a_1^2 - 1}{(a_1 - a_2)(1 - a_1 z)(a_1 - z)}.$$

The sum of these two contributions gives the right hand side of (3.8).

If  $a_1 > 1$  and  $a_2 < -1$ , the continuous part contributes

$$(3.9) \quad \frac{K(1/a_1, 1/a_2, z, 0)}{a_1^2 a_2^2 K(a_1, a_2, 0, 0)} = -\frac{1}{(z - a_1)(z - a_2)}.$$

The discrete part contributes

$$\begin{aligned} \frac{w(a_1, a_2)}{1 + z^2 - 2zy(a_1)} + \frac{w(a_2, a_1)}{1 + z^2 - 2zy(a_2)} &= \frac{w(a_1, a_2)}{(1 - a_1 z)(1 - z/a_1)} + \frac{w(a_2, a_1)}{(1 - a_2 z)(1 - z/a_2)} \\ &= \frac{(1 + a_1 a_2)(z^2 + 1) - 2(a_1 + a_2)z}{(z - a_1)(1 - a_1 z)(z - a_2)(1 - a_2 z)} = \frac{1}{(1 - a_1 z)(1 - a_2 z)} + \frac{1}{(z - a_1)(z - a_2)}. \end{aligned}$$

The sum of this expression and (3.9) gives the right hand side of (3.8).

The remaining cases with  $a_1$  or  $a_2$  taking values  $\pm 1$  are the limits of the above.  $\square$

The following identity will be used to verify Chapman-Kolmogorov equations.

**Proposition 3.4.** *If  $a_1, a_2$  satisfy Assumption 3.2 then for all  $-1 < m < 1$ , and all Borel sets  $U$ ,*

$$(3.10) \quad \nu(U; ma_1, ma_2) = \int_{\mathbb{R}} \nu\left(U; m(x + \sqrt{x^2 - 1}), m(x - \sqrt{x^2 - 1})\right) \nu(dx; a_1, a_2).$$

A short proof uses the following  $H$ -transform.

**Lemma 3.5.** *A compactly supported probability measure  $\nu$  is determined uniquely by the function  $z \mapsto H(z) = \int (1 + z^2 - 2zy)^{-1} \nu(dy)$  for  $z$  in a neighborhood of 0.*

*Proof.* A compactly supported measure is determined uniquely by its moments. The  $k$ -th moment of  $\nu$  can be computed from the  $k$ -th derivative of  $H$  at  $z = 0$  and the moments of lower orders.  $\square$

*Proof of Proposition 3.4.* Applying (3.8) twice, the  $H$ -transform of the right hand side of (3.10) is

$$\begin{aligned} & \int \frac{1}{(1 - zm(x + \sqrt{x^2 - 1}))(1 - zm(x - \sqrt{x^2 - 1}))} \nu(dx; a_1, a_2) \\ &= \int \frac{1}{1 + (mz)^2 - 2(mz)x} \nu(dx; a_1, a_2) = \frac{1}{(1 - mza_1)(1 - mza_2)}. \end{aligned}$$

From (3.8) we see that this matches the  $H$ -transform of the left hand side of (3.10).  $\square$

**3.2. An auxiliary Markov process.** Next we define transition probabilities of a Markov process with state space  $\mathbb{R}$  and time  $T = (CD, \infty)$ , where  $C, D$  are either real such that  $CD \geq 0$  or complex conjugate.

For  $t \in (CD, \infty)$  we define probability measures

$$\mu_t(dy) = \nu\left(dy; \frac{C}{\sqrt{t}}, \frac{D}{\sqrt{t}}\right),$$

and for  $s < t$ ,  $s, t \in [CD, \infty)$  and any real  $x$  we define probability measures

$$\mu_{s,t}(x, dy) = \nu\left(dy; \sqrt{\frac{s}{t}}(x + \sqrt{x^2 - 1}), \sqrt{\frac{s}{t}}(x - \sqrt{x^2 - 1})\right).$$

Note that these measures are well defined: in each case the corresponding parameters  $a_1, a_2$  are either real or complex conjugates, and their product satisfies  $a_1 a_2 < 1$ .

We want to check that these measures form a Markov family, that is:

**Proposition 3.6.** For  $CD < s < t$ ,

$$(3.11) \quad \mu_t(dy) = \int_{\mathbb{R}} \mu_{s,t}(x, dy) \mu_s(dx).$$

For  $CD < s < t < u$  and real  $x$ ,

$$(3.12) \quad \mu_{s,u}(x, dz) = \int_{\mathbb{R}} \mu_{t,u}(y, dz) \mu_{s,t}(x, dy).$$

In addition, we have

$$(3.13) \quad \int_{\mathbb{R}} (1 + z^2 - 2zy)^{-1} \mu_{s,t}(x, dy) = \begin{cases} \frac{t}{t + sz^2 - 2\sqrt{st}zx} & \text{if } |z| < 1, \\ \frac{t}{tz^2 + s - 2\sqrt{st}zx} & \text{if } |z| > 1. \end{cases}$$

*Proof.* Formula (3.11) follows from (3.10) applied to  $a_1 = C/\sqrt{s}$ ,  $a_2 = D/\sqrt{s}$  and  $m = \sqrt{s/t}$ . Formula (3.12) follows from (3.10) applied to  $a_1 = \sqrt{\frac{s}{t}}(x + \sqrt{x^2 - 1})$ ,  $a_2 = \sqrt{\frac{s}{t}}(x - \sqrt{x^2 - 1})$  and  $m = \sqrt{t/u}$ . Formula (3.13) follows from (3.8) applied to  $z$  when  $|z| < 1$  or to  $1/z$  when  $|z| > 1$ .  $\square$

*Remark 3.7.* The construction works also for real  $C, D$  such that  $CD < 0$ , with time  $T = (0, \infty)$ .

## 4. PROOFS OF THE MAIN RESULTS

*Proof of Proposition 2.1.* Let  $C, D$  denote the roots of  $z^2 + \theta z + \tau = 0$ , so that  $\tau = CD$  and  $\theta = -(C + D)$ . Of course,  $C, D$  are either real or complex conjugate, so the Markov process  $(Y_t)_{t > \tau}$  from Proposition 3.6 is well defined.

For rational  $t > 0$  define

$$(4.1) \quad X_t = \theta + 2\sqrt{t + \tau}Y_{t+\tau}.$$

Then  $(X_t)_{t \in \mathbb{Q}_+}$  is a Markov process. From Proposition 3.3 we see that

$$\mathbb{E}(Y_t) = \frac{C + D}{2\sqrt{t}}, \quad \text{Var}(Y_t) = \frac{t - CD}{4t},$$

so  $E(X_t) = 0$  and  $E(X_t^2) = t$ .

From (3.13) with  $z$  replaced by  $z\sqrt{t + \tau}$ , we get

$$\mathbb{E}\left(\frac{1}{1 + z^2(t + \tau) - 2z\sqrt{t + \tau}Y_{t+\tau}} \middle| Y_{s+\tau}\right) = \frac{1}{1 + z^2(s + \tau) - 2z\sqrt{s + \tau}Y_{t+\tau}}$$

for all  $s < t$  such that  $t + \tau < 1/|z|^2$ . This shows that Proposition 2.2 holds over positive rational  $t$ . In particular, taking the derivative with respect to  $z$  at  $z = 0$  we see that  $\theta + M'_t(0) = X_t$  is a (square-integrable) martingale. Therefore  $X_t = \lim_{q \rightarrow t^+, q \in \mathbb{Q}} X_q$  exists almost surely, and defines a Markov process with right-continuous trajectories that have left limits, see (Kallenberg, 1997, Theorem 6.27). Of course, the transition probabilities of  $(X_t)$  are re-calculated from the transition probabilities of  $(Y_{t+\tau})$ , and  $X_0 = 0$  since  $\text{Var}(X_t) = t$  for rational  $t > 0$ . (Details of calculation of transition probabilities for  $(X_t)$  are omitted.)  $\square$

*Proof of Proposition 2.2.* We already saw that the result holds true for rational  $t$ . The general version follows by taking the limit.  $\square$

*Proof of Theorem 2.3.* Fix  $f \in \mathcal{A}_t$  such that  $f$  is analytic in the disk  $|u - \theta| < 5/4(r_t + 2\delta)$  and take  $h > 0$  small so enough the support of  $X_{t+h}$  is in the disk  $|u - \theta| < r_t + \delta$ . Let  $\gamma$  be a curve in the first disk that encloses the support of  $X_{t+h}$ , and let  $x$  be in the support of  $X_{t+h}$ . Substituting  $u = 1/z + \theta + (t + \tau + h)z$  in the Cauchy formula  $f(x) = \frac{1}{2\pi i} \oint_{\gamma} f(u)(u - x)^{-1} du$ , we get

$$(4.2) \quad f(x) = \frac{1}{2\pi i} \oint_{|z|=1/(r_t+\delta)} \frac{g_{t+h}(z)}{1 - z(x - \theta) + (t + h + \tau)z^2},$$

where

$$g_t(z) = \left( (t + \tau)z - \frac{1}{z} \right) f \left( \theta + (t + \tau)z + \frac{1}{z} \right),$$

and  $\gamma$  is the ellipse  $u(s) = \theta + (r_t + \delta)e^{-is} + \frac{t+h+\tau}{r_t+\delta}e^{is}$ . Here we observe that

$$|u(s) - \theta| \leq r_t + \delta + \frac{r_{t+h}^2}{4(r_t + \delta)} < \frac{5}{4}(r_t + \delta)$$

for  $h$  small enough, so  $f$  is analytic in a disk that contains  $\gamma$ . Also  $\gamma$  encloses the interval  $(\theta - r_t - \delta, \theta + r_t + \delta)$  which for small enough  $h \geq 0$  contains the support of



$X_{t+h}$ . Recall that  $r_t \geq 2\sqrt{t+\tau}$ . From (4.2) we see that by Proposition 2.2 applied with  $h > 0$  small enough so that  $t+h+\tau < (r_t+\delta)^2$ ,

$$\begin{aligned} L_t(f)(x) &= \lim_{h \rightarrow 0^+} \frac{1}{2\pi i} \oint_{|z|=1/(r_t+\delta)} \frac{(g_{t+h}(z) - g_t(z))/h}{1 - z(x-\theta) + (t+\tau)z^2} dz \\ &= \frac{1}{2\pi i} \oint_{|z|=1/(r_t+\delta)} \frac{1}{1 - z(x-\theta) + (t+\tau)z^2} \frac{\partial g_t(z)}{\partial t} dz. \end{aligned}$$

Differentiating (4.2) with respect to  $h$  at  $h = 0$  we get

$$\begin{aligned} \frac{1}{2\pi i} \oint_{|z|=1/(r_t+\delta)} \frac{1}{1 - z(x-\theta) + (t+\tau)z^2} \frac{\partial g_t(z)}{\partial t} dz \\ = \frac{1}{2\pi i} \oint_{|z|=1/(r_t+\delta)} \frac{z^2 g_t(z)}{(1 - z(x-\theta) + (t+\tau)z^2)^2} dz. \end{aligned}$$

So

$$(4.3) \quad L_t(f)(x) = \frac{1}{2\pi i} \oint_{|z|=1/(r_t+\delta)} \frac{z^2 g_t(z)}{(1 - z(x-\theta) + (t+\tau)z^2)^2} dz.$$

We now verify that the right hand side of (2.5) gives the same answer. From (4.2) with  $h = 0$  we see that for  $x, y$  in the support of  $X_t$ ,

$$(4.4) \quad \frac{f(y) - f(x)}{y - x} = \frac{1}{2\pi i} \oint_{|z|=1/(r_t+\delta)} \frac{z g_t(z) dz}{(1 - z(x-\theta) + (t+\tau)z^2)(1 - z(y-\theta) + (t+\tau)z^2)}.$$

Now we note that the support of the semicircle law  $w_{\theta, t+\tau}$  is contained in the support of  $X_t$ , and that with  $u = \sqrt{t+\tau}z$  in the unit circle, by Proposition 2.2 applied to the case of semicircle law, i.e., to  $\theta = \tau = 0$  we have

$$\int_{\mathbb{R}} \frac{1}{1 - z(y-\theta) + (t+\tau)z^2} w_{\theta, t+\tau}(dy) = \int_{\mathbb{R}} \frac{1}{1 - uy + u^2} w_{0,1}(dy) = 1.$$

Thus integrating (4.4) we get

$$\int_{\mathbb{R}} \frac{f(y) - f(x)}{y - x} w_{\theta, t+\tau}(dy) = \frac{1}{2\pi i} \oint_{|z|=1/(r_t+\delta)} \frac{z g_t(z)}{1 - z(x-\theta) + (t+\tau)z^2} dz.$$

Taking the derivative of this expression with respect to  $x$  and using (4.3) we get (2.5).  $\square$

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## APPENDIX A. NOTES

**1.** Up to affine transformations, the univariate laws  $\{P_{0,t}(0, dy) : t > 0\}$  of  $(X_t)_{t>0}$  come from a two-parameter family of what is now called the "free Meixner laws". These laws were introduced as the orthogonality measures of systems of polynomials with constant recursions in Saitoh and Yoshida (2001) who found the explicit formula, analyzed free infinite divisibility and pointed out that this class includes a number of laws of interest in free probability; the term "free Meixner" was introduced in Anshelevich (2003). Further properties were studied in a series of papers Anshelevich (2004, 2005, 2007, 2008); Bożejko and Bryc (2006).

The free Meixner laws can be classified into six types: Wigner's semicircle (free Gaussian) which corresponds to our  $\tau = \theta = 0$ , free Poisson (also known as Marchenko-Pastur) which corresponds to our  $\tau = 0, \theta \neq 0$ , free Pascal (also known as free negative binomial) which corresponds to our  $\theta^2 > 4\tau > 0$ , free Gamma which corresponds to our  $\theta^2 = 4\tau > 0$ , a law that one may call pure free Meixner, and the free binomial law which corresponds to the case  $\tau < 0$  that is not considered in this note; the complete list of cases builds on (Saitoh and Yoshida, 2001, Remark 2.5 and Examples 3.4, 3.6), (Anshelevich, 2003, Theorem 4) and appears in (Bożejko and Bryc, 2006, Theorem 3.2) or in (Anshelevich, 2009, Remark 4).

**2.** The Markov process  $(X_t)$  can be introduced as follows. Except for the free binomial family, the free Meixner laws are infinitely-divisible with respect to the additive free convolution, see (Saitoh and Yoshida, 2001, Theorem 3.2), and are therefore univariate laws of non-commutative free Lévy processes. By Biane (1998,

Theorem 3.1) there exists a unique Markov process  $(X_t)$  with the same univariate laws and the same time-ordered joint moments (if they exist). In particular, if  $\theta = \tau = 0$ , the univariate laws of  $(X_t)$  are the semicircle laws  $w_{0,t}$ ,  $t > 0$ , and the corresponding transition probabilities appear in (Biane, 1998, Section 5.3).

The same family of Markov processes can be specified by conditional means and conditional variances, see (Bryc and Wołowski, 2005, Theorem 4.3), and our construction is based on the formulas from that paper.

**3.** Proposition 2.2 can be deduced from Biane (1998, Proposition 4.3.1). However to do so when  $\tau > 0$  one needs to use a non-trivial substitution that appears in (Anshelevich, 2003, page 236). Additional analysis is needed to determine explicitly the allowed range of  $t$  which is crucial for our proof of Theorem 2.3.

**4.** For  $\theta = \tau = 0$ , formula (2.5) agrees with the non-commutative result Biane and Speicher (1998, page 392) after correcting their expression by a factor of 2, and with (Bożejko et al., 1997, page 150), who consider a closely related classical Markov process  $(e^t X_{e^{-2t}})_{t>0}$  with the generator

$$L_t(f)(x) = xf'(x) - 2 \frac{\partial}{\partial x} \int \frac{f(y) - f(x)}{y - x} w_{0,1}(dy).$$

**5.** Generators of more general Markov processes that arise from free Lévy processes can be read out from (Anshelevich, 2002, Corollary 10). For properties of operator  $f \mapsto \int \frac{f(y) - f(x)}{y - x} \mu(dy)$  with compactly supported  $\mu$  see (Anshelevich, 2009, Proposition 1).

**6.** Combining Proposition 2.2 with Lemma 3.5, and (Biane, 1998, Proposition 4.3.1) with (Anshelevich, 2003, page 236) one verifies that the action of transition probabilities of  $(X_t)$  on polynomials  $f$  coincides with the action of non-commutative conditional expectation, so the joint moments of our process  $(X_t)$  indeed match the non-commutative moments as explained in (Biane, 1998, page 161)

**7.** Lemma 3.1 is an elementary case of the Askey-Wilson integral (Askey and Wilson, 1985, (2.1)). Ismail and Masson (1995, Eqn (1.3)) state this elementary integral when  $a_3 = a_4 = 0$ .

**8.** A version of Lemma 3.5 holds true also for non-compactly supported measures, as  $H(z) = G((z+1/z)/2)/(2z)$ , where  $G(u) = \int (u-x)\nu(dx)$  is the Cauchy-Stieltjes transform of  $\nu$ .

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